DAY TWENTY TWO

Inverse Trigonometric Function

Learning & Revision for the Day

- Inverse Trigonometric Function
- Properties of Inverse Trigonometric Function

Inverse Trigonometric Function

Trigonometric functions are not one-one and onto on their natural domains and ranges, so their inverse do not exists in the whole domain. If we restrict their domain and range, then their inverse may exists.

$y = f(x) = \sin x$. Then, its inverse is $x = \sin^{-1} y$.

NOTE
- $\sin^{-1} y \neq (\sin y)^{-1}$
- $\sin^{-1} y \neq \sin\left(\frac{1}{y}\right)$

The value of an inverse trigonometric functions which lies in its principal value branch is called the principal value of that inverse trigonometric function.

Domain and range of inverse trigonometric functions

Function	Domain	Range (Principal Value Branch)
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	R	$(0, \pi)$
$\sec^{-1} x$	$R-(-1, 1)$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$
$\text{cosec}^{-1} x$	$R-(-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-\{0\}$

Properties of Inverse Trigonometric Functions

1. (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2};$ $(-1 \le x \le 1)$

   (ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2};$ $x \in R$

   (iii) $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2};$ $(x \le -1 \text{ or } x \ge 1)$

2. (i) $\sin^{-1}(-x) = -\sin^{-1} x;$ $(-1 \le x \le 1)$

   (ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x;$ $(-1 \le x \le 1)$

   (iii) $\tan^{-1}(-x) = -\tan^{-1}(x);$ $(-\infty < x < \infty)$

   (iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x;$ $(-\infty < x < \infty)$

   (v) $\sec^{-1}(-x) = \pi - \sec^{-1} x;$ $x \le -1 \text{ or } x \ge 1)$

   (vi) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x;$ $(x \le -1 \text{ or } x \ge 1)$

3. (i) $\sin^{-1}(\sin x)$ is a periodic function with period 2π.

$$\sin^{-1}(\sin x) = \begin{cases} x, & x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \pi - x, & x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ x - 2\pi, & x \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \\ 3\pi - x, & x \in \left[\frac{5\pi}{2}, \frac{7\pi}{2}\right] \end{cases}$$

   (ii) $\cos^{-1}(\cos x)$ is a periodic function with period 2π.

$$\cos^{-1}(\cos x) = \begin{cases} x, & x \in [0, \pi] \\ 2\pi - x, & x \in [\pi, 2\pi] \\ x - 2\pi, & x \in [2\pi, 3\pi] \\ 4\pi - x, & x \in [3\pi, 4\pi] \end{cases}$$

   (iii) $\tan^{-1}(\tan x)$ is a periodic function with period π.

$$\tan^{-1}(\tan x) = \begin{cases} x, & x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ x - \pi, & x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ x - 2\pi, & x \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \\ x - 3\pi, & x \in \left[\frac{5\pi}{2}, \frac{7\pi}{2}\right] \end{cases}$$

   (iv) $\cot^{-1}(\cot x)$ is a periodic function with period π.
   $\cot^{-1}(\cot x) = x;\ \ 0 < x < \pi$

   (v) $\sec^{-1}(\sec x)$ is a periodic function with period 2π.
   $\sec^{-1}(\sec x) = x;\ 0 \le x < \frac{\pi}{2} \text{ or } \frac{\pi}{2} < x \le \pi$

   (vi) $\text{cosec}^{-1}(\text{cosec } x)$ is a periodic function with period 2π.
   $\text{cosec}^{-1}(\text{cosec } x) = x; -\frac{\pi}{2} \le x < 0 \text{ or } 0 < x \le \frac{\pi}{2}$

4. (i) $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}^{-1} x, \text{ if } x \in (-\infty, -1] \cup [1, \infty)$

   (ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x, \text{ if } x \in (-\infty, -1] \cup [1, \infty)$

   (iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{if } x > 0 \\ -\pi + \cot^{-1} x, & \text{if } x < 0 \end{cases}$

5. (i) $\sin^{-1} x = \cos^{-1}\sqrt{1 - x^2}$

$$= \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$= \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \text{cosex}^{-1}\left(\frac{1}{x}\right), \text{ if } x \in (0,1)$$

   (ii) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$

$$= \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\left(\frac{1}{x}\right)$$

$$= \text{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right), \text{ if } x \in (0,1)$$

   (iii) $\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

$$= \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\frac{1}{x}\right)$$

$$= \text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right) = \sec^{-1}(\sqrt{1+x^2}), \text{ if } x \in (0, \infty)$$

6. (i) $\sin^{-1} x + \sin^{-1} y$

$$= \begin{cases} \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}); \ |x|, |y| \le 1 \text{ and} \\ \quad x^2 + y^2 \le 1 \text{ or } (xy < 0 \text{ and } x^2 + y^2 > 1) \\ \pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}); \ 0 < x, y \le 1 \\ \quad \text{and } \ x^2 + y^2 > 1 \\ -\pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}); \\ -1 \le x, y < 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

   (ii) $\sin^{-1} x - \sin^{-1} y$

$$= \begin{cases} \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2}); \ |x|, |y| \le 1 \\ \text{and } x^2 + y^2 \le 1 \text{ or } (xy > 0 \text{ and } x^2 + y^2 > 1) \\ \pi - \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2}); \\ 0 < x \le 1, -1 \le y < 0 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2}); \\ -1 \le x < 0, 0 < y \le 1 \ \text{ and } x^2 + y^2 > 1 \end{cases}$$

   (iii) $\cos^{-1} x + \cos^{-1} y$

$$= \begin{cases} \cos^{-1}\{xy - \sqrt{(1-x^2)}\sqrt{(1-y^2)}\}; \ |x|, |y| \le 1 \\ \quad \text{and } x + y \ge 0 \\ 2\pi - \cos^{-1}\{xy - \sqrt{(1-x^2)}\sqrt{(1-y^2)}\}; \\ \quad |x|, |y| \le 1 \text{ and } x + y \le 0 \end{cases}$$

   (iv) $\cos^{-1} x - \cos^{-1} y$

$$= \begin{cases} \cos^{-1}\{xy + \sqrt{(1-x^2)}\sqrt{(1-y^2)}\}; |x|, |y| \le 1 \\ \text{and } x \le y \\ -\cos^{-1}\{xy + \sqrt{(1-x^2)}\sqrt{(1-y^2)}\}; \\ -1 \le y \le 0, 0 < x \le 1 \text{ and } x \ge y \end{cases}$$

(v) $\tan^{-1} x + \tan^{-1} y$

$$= \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & x > 0, y > 0, xy > 1 \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right); & x < 0, y < 0, xy > 1 \end{cases}$$

(vi) $\tan^{-1} x - \tan^{-1} y$

$$= \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right); & xy > -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right); & xy < -1, x > 0, y < 0 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right); & xy < -1, x < 0, y > 0 \end{cases}$$

7. (i) $2 \sin^{-1} x$

$$= \begin{cases} \sin^{-1}\{2x\sqrt{(1-x^2)}\}; & -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}); & \dfrac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}); & -1 \le x < -\dfrac{1}{\sqrt{2}} \end{cases}$$

(ii) $2\cos^{-1} x = \begin{cases} \cos^{-1}(2x^2 - 1); & 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2 - 1); & -1 \le x < 0 \end{cases}$

(iii) $2\tan^{-1} x = \begin{cases} \tan^{-1}\left(\dfrac{2x}{1-x^2}\right); & -1 < x < 1 \\ \pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right); & x > 1 \\ -\pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right); & x < -1 \end{cases}$

(iv) $2\tan^{-1} x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & x > 1 \\ -\pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & x < -1 \end{cases}$

(v) $2\tan^{-1} x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); & 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); & -\infty < x \le 0 \end{cases}$

NOTE
- If $\sin^{-1} x + \sin^{-1} y = \theta$, then $\cos^{-1} x + \cos^{-1} y = \pi - \theta$
- If $\cos^{-1} x + \cos^{-1} y = \theta$, then $\sin^{-1} x + \sin^{-1} y = \pi - \theta$

8. (i) $3\sin^{-1} x = \begin{cases} \sin^{-1}(3x - 4x^3), & \text{if } \dfrac{-1}{2} \le x \le \dfrac{1}{2} \\ \pi - \sin^{-1}(3x - 4x^3), & \text{if } \dfrac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x - 4x^3), & \text{if } -1 \le x < \dfrac{-1}{2} \end{cases}$

(ii) $3\cos^{-1} x = \begin{cases} \cos^{-1}(4x^3 - 3x), & \text{if } \dfrac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3 - 3x), & \text{if } \dfrac{-1}{2} \le x \le \dfrac{1}{2} \\ 2\pi + \cos^{-1}(4x^3 - 3x), & \text{if } -1 \le x \le \dfrac{1}{2} \end{cases}$

(iii) $3\tan^{-1} x = \begin{cases} \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right), & \text{if } \dfrac{-1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right), & \text{if } x > \dfrac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\dfrac{3x - x^3}{1 - 3x^2}\right), & \text{if } x < \dfrac{-1}{\sqrt{3}} \end{cases}$

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 The principal value of $\sin^{-1}\left(\cos\dfrac{33\pi}{5}\right)$ is → **NCERT Exemplar**

(a) $\dfrac{3\pi}{5}$ (b) $\dfrac{7\pi}{5}$ (c) $\dfrac{\pi}{10}$ (d) $-\dfrac{\pi}{10}$

2 If $\sum_{i=1}^{20} \sin^{-1} x_i = 10\pi$, then $\sum_{i=1}^{20} x_i$ is equal to

(a) 20 (b) 10 (c) 0 (d) None of these

3 The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is → **NCERT Exemplar**

(a) [1, 2] (b) [– 1, 1]
(c) [0, 1] (d) None of these

4 The value of $\cos(2\cos^{-1} x + \sin^{-1} x)$ at $x = \dfrac{1}{5}$ is

(a) 1 (b) 3 (c) 0 (d) $-\dfrac{2\sqrt{6}}{5}$

5 The value of $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}]$ is

(a) $\dfrac{1}{\sqrt{x^2+2}}$ (b) $\sqrt{\dfrac{x^2+2}{x^2+1}}$ (c) $\sqrt{\dfrac{x^2+1}{x^2+2}}$ (d) $\dfrac{1}{\sqrt{x^2+1}}$

6 The equation $\tan^{-1} x - \cot^{-1} x = \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ has → **NCERT Exemplar**

(a) no solution
(b) unique solution
(c) infinite number of solutions
(d) two solutions

7 Let $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x| < \frac{1}{\sqrt{3}}$. Then, a value of y is → **JEE Mains 2015**

(a) $\frac{3x - x^3}{1 - 3x^2}$ (b) $\frac{3x + x^3}{1 - 3x^2}$

(c) $\frac{3x - x^3}{1 + 3x^2}$ (d) $\frac{3x + x^3}{1 + 3x^2}$

8 If $\theta = \tan^{-1} a$, $\phi = \tan^{-1} b$ and $ab = -1$, then $(\theta - \phi)$ is equal to

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) None of these

9 The range of

$f(x) = |3 \tan^{-1} x - \cos^{-1}(0)| - \cos^{-1}(-1)$ is

(a) $[-\pi, \pi)$ (b) $(-\pi, \pi)$ (c) $[-\pi, \pi]$ (d) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$

10 The number of solutions of the equation $\cos(\cos^{-1} x) = \operatorname{cosec}(\operatorname{cosec}^{-1} x)$ is

(a) 2 (b) 3 (c) 4 (d) 1

11 The value of $\cot\left(\operatorname{cosec}^{-1}\frac{5}{3} + \tan^{-1}\frac{2}{3}\right)$ is → **AIEEE 2008**

(a) $\frac{5}{17}$ (b) $\frac{6}{17}$ (c) $\frac{3}{17}$ (d) $\frac{4}{17}$

12 If $\tan(\cos^{-1} x) = \sin\left(\cot^{-1}\frac{1}{2}\right)$, then x is equal to

(a) $\pm\frac{5}{3}$ (b) $\pm\frac{\sqrt{5}}{3}$ (c) $\pm\frac{5}{\sqrt{3}}$ (d) None of these

13 If $\cos^{-1} x + \cos^{-1} y = \frac{\pi}{2}$ and $\tan^{-1} x - \tan^{-1} y = 0$, then $x^2 + xy + y^2$ is equal to

(a) 0 (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{3}{2}$ (d) $\frac{1}{8}$

14 If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then the value of x is → **AIEEE 2007**

(a) 1 (b) 3 (c) 4 (d) 5

15 If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$, then the value of

$\sum \frac{(x^{101} + y^{101})(x^{202} + y^{202})}{(x^{303} + y^{303})(x^{404} + y^{404})}$ is

(a) 0 (b) 1 (c) 2 (d) 3

16 The root of the equation

$\tan^{-1}\left(\frac{x-1}{x+1}\right) + \tan^{-1}\left(\frac{2x-1}{2x+1}\right) = \tan^{-1}\left(\frac{23}{36}\right)$ is

(a) $-\frac{3}{8}$ (b) $-\frac{1}{2}$ (c) $\frac{3}{4}$ (d) $\frac{4}{3}$

17 The number of solutions of

$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2 + x + 1} = \frac{\pi}{2}$ is

→ **NCERT Exemplar**

(a) 0 (b) 1 (c) 2 (d) infinite

18 The maximum value of $(\sec^{-1} x)^2 + (\operatorname{cosec}^{-1} x)^2$ is

(a) $\frac{\pi^2}{2}$ (b) $\frac{5\pi^2}{4}$

(c) π^2 (d) None of these

19 The trigonometric equation $\sin^{-1} x = 2\sin^{-1} a$, has a solution for → **AIEEE 2003**

(a) $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$ (b) all real values of a

(c) $|a| \le \frac{1}{\sqrt{2}}$ (d) $|a| \ge \frac{1}{\sqrt{2}}$

20 If $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then the value of

$\tan^{-1}\left(\frac{\tan x}{4}\right) + \tan^{-1}\left(\frac{3\sin 2x}{5 + 3\cos 2x}\right)$ is

(a) $\frac{x}{2}$ (b) $2x$ (c) $3x$ (d) x

21 If $\tan\theta + \tan\left(\frac{\pi}{3} + \theta\right) + \tan\left(-\frac{\pi}{3} + \theta\right) = a\tan 3\theta$, then a is equal to

(a) 1/3 (b) 1

(c) 3 (d) None of these

22 If $\cos^{-1} x = \tan^{-1} x$, then $\sin(\cos^{-1} x)$ is equal to

(a) $-x$ (b) x^2 (c) x^3 (d) $-\frac{1}{x^2}$

23 The real solution of

$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2 + x + 1} = \frac{\pi}{2}$ is

(a) 2, 3 (b) 1, 0 (c) −1, 0 (d) 3, 1

24 If $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then the value of x is → **AIEEE 2007**

(a) 1 (b) 3 (c) 4 (d) 5

25 If $\operatorname{cosec}^{-1} x + \cos^{-1} y + \sec^{-1} z \ge \alpha^2 - \sqrt{2\pi}\,\alpha + 3\pi$, where α is a real number, then

(a) $x = 1, y = -1$ (b) $x = -1, z = -1$

(c) $x = 2, y = 1$ (d) $x = 1, y = -2$

26 The solution of $\sin^{-1} x \le \cos^{-1} x$ is

(a) $\left(-1, \frac{1}{\sqrt{2}}\right)$ (b) $\left[-1, \frac{1}{\sqrt{2}}\right]$

(c) $\left[1, \frac{1}{\sqrt{2}}\right]$ (d) $\left(1, \frac{1}{\sqrt{2}}\right)$

27 If m and M are the least and the greatest value of $(\cos^{-1} x)^2 + (\sin^{-1} x)^2$, then $\frac{M}{m}$ is equal to

(a) 10 (b) 5 (c) 4 (d) 2

28 The number of real solutions of the equation $\sqrt{1 + \cos 2x} = \sqrt{2}\cos^{-1}(\cos x)$ in $\left[\frac{\pi}{2}, \pi\right]$ is

→ **NCERT Exemplar**

(a) 0 (b) 1 (c) 2 (d) infinite

29 The sum of the infinite series

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+\sin^{-1}\left(\frac{\sqrt{2}-1}{\sqrt{6}}\right)+\sin^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{\sqrt{12}}\right)+\ldots+\sin^{-1}\left(\frac{\sqrt{n}-\sqrt{(n-1)}}{\sqrt{n(n+1)}}\right)$$ is

(a) $\frac{\pi}{8}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π

30 A root of the equation

$17x^2+17x\tan\left[2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right]-10=0$ is

(a) $\frac{10}{17}$ (b) -1 (c) $-\frac{7}{17}$ (d) 1

31 The value of x for which $\sin[\cot^{-1}(1+x)]=\cos(\tan^{-1}x)$, is **➜ JEE Mains 2013**

(a) $-\frac{1}{2}$ (b) 1 (c) 0 (d) $\frac{1}{2}$

32 If $0<x<1$, then $\sqrt{1+x^2}[\{x\cos(\cot^{-1}x)+\sin(\cot^{-1}x)\}^2-1]^{1/2}$ is equal to

(a) $\frac{x}{\sqrt{1+x^2}}$ (b) x (c) $x\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$

33 If x, y and z are in AP and $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are also in AP, then **➜ AIEEE 2012**

(a) $x=y=z$ (b) $2x=y=6z$
(c) $6x=3y=2z$ (d) $6x=4y=3z$

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 If $\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}-\ldots\right)+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-\ldots\right)=\frac{\pi}{2}$, where $0<|x|<\sqrt{2}$, then x is equal to

(a) $\frac{1}{2}$ (b) 1 (c) $-\frac{1}{2}$ (d) -1

2 If the mapping $f(x)=ax+b$, $a>0$ maps $[-1,1]$ onto $[0,2]$, then $\cot[\cot^{-1}7+\cot^{-1}8+\cot^{-1}18]$ is equal to

(a) $f(-1)$ (b) $f(0)$ (c) $f(1)$ (d) $f(2)$

3 If $S=\tan^{-1}\left(\frac{1}{n^2+n+1}\right)+\tan^{-1}\left(\frac{1}{n^2+3n+3}\right)+\ldots+\tan^{-1}\left\{\frac{1}{1+(n+19)(n+20)}\right\}$, then $\tan S$ is equal to **➜ JEE Mains 2013**

(a) $\frac{20}{401+20n}$ (b) $\frac{n}{n^2+20n+1}$
(c) $\frac{20}{n^2+20n+1}$ (d) $\frac{n}{401+20n}$

4 If $f(x)=e^{\cos^{-1}\sin\left(x+\frac{\pi}{3}\right)}$, then

(a) $f\left(-\frac{7\pi}{4}\right)=e^{\frac{\pi}{11}}$ (b) $f\left(\frac{8\pi}{9}\right)=e^{\frac{13\pi}{18}}$
(c) $f\left(-\frac{7\pi}{4}\right)=e^{\frac{3\pi}{12}}$ (d) $f\left(-\frac{7\pi}{4}\right)=e^{\frac{11\pi}{13}}$

5 If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$ and $f(1)=2$, $f(p+q)=f(p)\cdot f(q)$, $\forall\ p,q\in R$, then $x^{f(1)}+y^{f(2)}+z^{f(3)}-\frac{(x+y+z)}{x^{f(1)}+y^{f(2)}+z^{f(3)}}$ is equal to

(a) 0 (b) 1
(c) 2 (d) 3

6 If $[\cot^{-1}x]+[\cos^{-1}x]=0$, where x is a non-negative real number and [.] denotes the greatest integer function, then complete set of values of x is

(a) $(\cos 1, 1]$ (b) $(\cot 1, 1)$
(c) $(\cos 1, \cot 1)$ (d) None of these

7 $\cot^{-1}(\sqrt{\cos\alpha})-\tan^{-1}(\sqrt{\cos\alpha})=x$, then $\sin x$ is equal to **➜ AIEEE 2002**

(a) $\tan^2\left(\frac{\alpha}{2}\right)$ (b) $\cot^2\left(\frac{\alpha}{2}\right)$ (c) $\tan\alpha$ (d) $\cot\left(\frac{\alpha}{2}\right)$

8 If $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$, then the value of x is

(a) -2 (b) -3 (c) -1 (d) 2

9 The solution set of $\tan^2(\sin^{-1}x)>1$ is

(a) $\left(-1,-\frac{1}{\sqrt{2}}\right)\cup\left(\frac{1}{\sqrt{2}},1\right)$ (b) $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\sim\{0\}$
(c) $(-1,1)\sim\{0\}$ (d) None of these

10 If θ and ϕ are the roots of the equation $8x^2+22x+5=0$, then

(a) both $\sin^{-1}\theta$ and $\sin^{-1}\phi$ are equal
(b) both $\sec^{-1}\theta$ and $\sec^{-1}\phi$ are real
(c) both $\tan^{-1}\theta$ and $\tan^{-1}\phi$ are real
(d) None of the above

11 $2\tan^{-1}(-2)$ is equal to

(a) $\cos^{-1}\left(\frac{-3}{5}\right)$ (b) $\pi+\cos^{-1}\frac{3}{5}$
(c) $-\frac{\pi}{2}+\tan^{-1}\left(-\frac{3}{4}\right)$ (d) $-\pi+\cot^{-1}\left(-\frac{3}{4}\right)$

12 Let $x\in(0,1)$. The set of all x such that $\sin^{-1}x>\cos^{-1}x$, is the interval **➜ JEE Mains 2013**

(a) $\left(\frac{1}{2},\frac{1}{\sqrt{2}}\right)$ (b) $\left(\frac{1}{\sqrt{2}},1\right)$
(c) $(0,1)$ (d) $\left(0,\frac{\sqrt{3}}{2}\right)$

ANSWERS

SESSION 1	**1** (d)	**2** (a)	**3** (a)	**4** (d)	**5** (c)	**6** (b)	**7** (a)	**8** (c)	**9** (a)	**10** (a)
	11 (b)	**12** (b)	**13** (c)	**14** (b)	**15** (d)	**16** (d)	**17** (b)	**18** (b)	**19** (c)	**20** (d)
	21 (c)	**22** (b)	**23** (c)	**24** (b)	**25** (a)	**26** (b)	**27** (a)	**28** (a)	**29** (c)	**30** (d)
	31 (a)	**32** (c)	**33** (a)							
SESSION 2	**1** (b)	**2** (d)	**3** (c)	**4** (b)	**5** (c)	**6** (b)	**7** (a)	**8** (c)	**9** (a)	**10** (c)
	11 (c)	**12** (b)								

Hints and Explanations

SESSION 1

1 $\cos\left(\frac{33\pi}{5}\right) = \cos\left(6\pi + \frac{3\pi}{5}\right) = \cos\frac{3\pi}{5}$

$= \sin\left(\frac{\pi}{2} - \frac{3\pi}{5}\right) = \sin\left(-\frac{\pi}{10}\right)$

$\therefore \quad \sin^{-1}\left(\cos\frac{33\pi}{5}\right)$

$= \sin^{-1}\left(\sin\left(-\frac{\pi}{10}\right)\right) = -\frac{\pi}{10}$

2 Since, $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$\therefore \quad \sin^{-1} x_i = \frac{\pi}{2}, \quad 1 \le i \le 20$

$\Rightarrow \quad x_i = 1, 1 \le i \le 20$

Thus, $\sum_{i=1}^{20} x_i = 20$

3 Given, $f(x) = \sin^{-1}\sqrt{x-1}$

For domain of $f(x) \qquad -1 \le \sqrt{x-1} \le 1$

$\Rightarrow \quad 0 \le (x-1) \le 1 \Rightarrow 1 \le x \le 2$

$\therefore \qquad x \in [1, 2]$

4 $\cos(2\cos^{-1} x + \sin^{-1} x)$

$= \cos[2(\cos^{-1} x + \sin^{-1} x) - \sin^{-1} x]$

$= \cos(\pi - \sin^{-1} x) = -\cos(\sin^{-1} x)$

$= -\cos\left[\sin^{-1}\left(\frac{1}{5}\right)\right] \qquad \left[\because x = \frac{1}{5}\right]$

$= -\cos\left[\cos^{-1}\sqrt{1-\left(\frac{1}{5}\right)^2}\right]$

$= -\cos\left(\cos^{-1}\frac{2\sqrt{6}}{5}\right) = -\frac{2\sqrt{6}}{5}$

5 We have, $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}]$

Let $\cot^{-1} x = \alpha$

$\Rightarrow \quad \cot\alpha = x$

$\Rightarrow \quad \operatorname{cosec}\alpha = \sqrt{1+x^2}$

$\Rightarrow \quad \sin\alpha = \frac{1}{\sqrt{1+x^2}}$

$\Rightarrow \quad \alpha = \sin^{-1}\frac{1}{\sqrt{1+x^2}}$

Hence, $\cos[\tan^{-1}\{\sin(\cot^{-1} x)\}]$

$= \cos\left[\tan^{-1}\left\{\sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right\}\right]$

$= \cos\left[\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right]$

$= \cos\left[\cos^{-1}\sqrt{\frac{x^2+1}{x^2+2}}\right]$

$\left[\because \text{let } \tan^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \beta\right.$

$\tan\beta = \frac{1}{\sqrt{1+x^2}}$

$\sec\beta = \sqrt{1+\frac{1}{1+x^2}} = \sqrt{\frac{x^2+2}{x^2+1}}$

$\left.\cos\beta = \sqrt{\frac{x^2+1}{x^2+2}}\right]$

$= \sqrt{\frac{x^2+1}{x^2+2}}$

6 Given, $\tan^{-1} x - \cot^{-1} x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\Rightarrow \tan^{-1} x - \cot^{-1} x = \frac{\pi}{6} \qquad \ldots(i)$

But $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \qquad \ldots(ii)$

On adding Eqs. (i) and (ii), we get

$2\tan^{-1} x = \frac{\pi}{6} + \frac{\pi}{2} = \frac{2\pi}{3}$

$\Rightarrow \tan^{-1} x = \frac{\pi}{3}$

$\Rightarrow \qquad x = \tan\frac{\pi}{3} \Rightarrow x = \sqrt{3}$

It has unique solution.

7 Given,

$\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$,

where $|x| < \frac{1}{\sqrt{3}}$

$\Rightarrow \tan^{-1} y = \tan^{-1}\left\{\frac{x + \frac{2x}{1-x^2}}{1 - x\left(\frac{2x}{1-x^2}\right)}\right\}$

$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \quad xy < 1\right]$

$= \tan^{-1}\left(\frac{x - x^3 + 2x}{1 - x^2 - 2x^2}\right)$

$\tan^{-1} y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

$\Rightarrow \qquad y = \frac{3x - x^3}{1 - 3x^2}$

Alternate Method

$|x| < \frac{1}{\sqrt{3}} \Rightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Let $x = \tan\theta \Rightarrow -\frac{\pi}{6} < \theta < \frac{\pi}{6}$

$\therefore \quad \tan^{-1} y = \theta + \tan^{-1}(\tan 2\theta)$

$= \theta + 2\theta = 3\theta$

$\Rightarrow \qquad y = \tan 3\theta$

$\Rightarrow \qquad y = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan\theta}$

$\Rightarrow \qquad y = \frac{3x - x^3}{1 - 3x^2}$

8 Given that, $\theta = \tan^{-1} a$

and $\qquad \phi = \tan^{-1} b$

and $\qquad ab = -1$

$\therefore \tan\theta\tan\phi = ab = -1$

$$\Rightarrow \quad \tan\theta = -\cot\phi$$

$$\Rightarrow \quad \tan\theta = \tan\left(\frac{\pi}{2} + \phi\right)$$

$$\Rightarrow \quad \theta - \phi = \frac{\pi}{2}$$

9 $f(x) = |3\tan^{-1} x - \cos^{-1}(0)| - \cos^{-1}(-1)$

$$= \left|3\tan^{-1} x - \left(\frac{\pi}{2}\right)\right| - \pi$$

We know that, $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$

$$\Rightarrow \quad \frac{-3\pi}{2} < 3\tan^{-1} x < \frac{3\pi}{2}$$

$$\Rightarrow \quad -2\pi < 3\tan^{-1} x - \frac{\pi}{2} < \pi$$

$$\Rightarrow \quad 0 \le \left|3\tan^{-1} x - \frac{\pi}{2}\right| < 2\pi$$

$$\Rightarrow \quad -\pi \le \left|3\tan^{-1} x - \frac{\pi}{2}\right| - \pi < \pi$$

10 $\cos(\cos^{-1} x) = x,\ \forall\ x \in [-1, 1]$ and $\operatorname{cosec}(\operatorname{cosec}^{-1} x)$

$$= x,\ \forall\ x \in (-\infty, -1] \cup [1, \infty)$$

$\Rightarrow$ $\cos(\cos^{-1} x) = \operatorname{cosec}(\operatorname{cosec}^{-1} x)$ for $x = \pm 1$ only.

Hence, there are two roots.

11 Since, $\operatorname{cosec}^{-1}\left(\frac{5}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

$$\therefore \cot\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right)$$

$$= \cot\left(\tan^{-1}\left[\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{1}{2}}\right]\right)$$

$$= \cot\left(\tan^{-1}\left[\frac{\left(\frac{17}{12}\right)}{\left(\frac{1}{2}\right)}\right]\right)$$

$$= \cot\left[\tan^{-1}\left(\frac{17}{6}\right)\right] = \frac{6}{17}$$

12 Let $\cot^{-1}\frac{1}{2} = \phi \Rightarrow \frac{1}{2} = \cot\phi$

$$\Rightarrow \quad \sin\phi = \frac{1}{\sqrt{1+\cot^2\phi}} = \frac{2}{\sqrt{5}}$$

Let $\cos^{-1} x = \theta \Rightarrow \sec\theta = \frac{1}{x}$

$$\Rightarrow \quad \tan\theta = \sqrt{\sec^2\theta - 1}$$

$$\Rightarrow \quad \tan\theta = \sqrt{\frac{1}{x^2} - 1} \Rightarrow \tan\theta = \frac{\sqrt{1-x^2}}{x}$$

Now, $\tan(\cos^{-1} x) = \sin\left(\cot^{-1}\frac{1}{2}\right)$

$$\Rightarrow \quad \tan\left(\tan^{-1}\frac{\sqrt{1-x^2}}{x}\right) = \sin\left(\sin^{-1}\frac{2}{\sqrt{5}}\right)$$

$$\Rightarrow \quad \frac{\sqrt{1-x^2}}{x} = \frac{2}{\sqrt{5}} \Rightarrow \sqrt{(1-x^2)5} = 2x$$

On squaring both sides, we get

$$(1-x^2)5 = 4x^2$$

$$\Rightarrow \quad 9x^2 = 5 \Rightarrow x = \pm\frac{\sqrt{5}}{3}$$

13 $\because \tan^{-1} x - \tan^{-1} y = 0 \Rightarrow x = y$

Also, $\cos^{-1} x + \cos^{-1} y = \frac{\pi}{2}$

$$\Rightarrow \quad 2\cos^{-1} x = \frac{\pi}{2} \Rightarrow \cos^{-1} x = \frac{\pi}{4}$$

$$\Rightarrow \quad x = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad x^2 = \frac{1}{2}$$

Hence, $x^2 + xy + y^2 = 3x^2 = \frac{3}{2}$

14 $\because \sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$

$$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{4}{5}\right)$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) = \cos^{-1}\left(\frac{4}{5}\right)$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right)$$

$$\therefore \quad x = 3$$

15 Given, $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$

$$\Rightarrow \quad \sin^{-1} x = \sin^{-1} y = \sin^{-1} z = \frac{\pi}{2}$$

$$\Rightarrow \quad x = y = z = 1$$

$$\therefore \Sigma\frac{(x^{101} + y^{101})(x^{202} + y^{202})}{(x^{303} + y^{303})(x^{404} + y^{404})} = \Sigma\frac{(1+1)(1+1)}{(1+1)(1+1)} = \Sigma 1 = 3$$

16 $\tan^{-1}\left[\dfrac{\frac{x-1}{x+1} + \frac{2x-1}{2x+1}}{1 - \left(\frac{x-1}{x+1}\right)\left(\frac{2x-1}{2x+1}\right)}\right] = \tan^{-1}\left(\frac{23}{36}\right)$

$$\Rightarrow \quad \frac{2x^2 - 1}{3x} = \frac{23}{36}$$

$$\Rightarrow \quad 24x^2 - 12 - 23x = 0$$

$$\Rightarrow \quad x = \frac{4}{3}, -\frac{3}{8}$$

But x cannot be negative.

$$\therefore \quad x = \frac{4}{3}$$

17 For existence, $x(x+1) \ge 0$ ...(i)

and $x^2 + x + 1 \le 1$

$\Rightarrow$ $x^2 + x \le 0$

$\Rightarrow$ $x(x+1) \le 0$ ...(ii)

From Eqs. (i) and (ii), we get

$x(x+1) = 0 \Rightarrow x = 0, -1$

But $x = -1$ is not satisfied the given equation.

18 Let $I = (\sec^{-1} x)^2 + (\operatorname{cosec}^{-1} x)^2$

$$= (\sec^{-1} x + \operatorname{cosec}^{-1} x)^2 - 2\sec^{-1} x \operatorname{cosec}^{-1} x$$

$$= \frac{\pi^2}{4} - 2\sec^{-1} x\left(\frac{\pi}{2} - \sec^{-1} x\right)$$

$$= \frac{\pi^2}{4} + 2\left[(\sec^{-1} x)^2 - \frac{\pi}{2}(\sec^{-1} x) + \frac{\pi^2}{16} - \frac{\pi^2}{16}\right]$$

$$= \frac{\pi^2}{8} + 2\left[\left(\sec^{-1} x - \frac{\pi}{4}\right)^2\right]$$

$$\therefore \quad I_{max} = \frac{\pi^2}{8} + 2\left[\frac{9\pi^2}{16}\right] = \frac{5\pi^2}{4}$$

19 Given that, $\sin^{-1} x = 2\sin^{-1} a$

$$\because \quad -\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$$

$$\Rightarrow \quad -\frac{\pi}{2} \le 2\sin^{-1} a \le \frac{\pi}{2}$$

$$\Rightarrow \quad -\frac{\pi}{4} \le \sin^{-1} a \le \frac{\pi}{4}$$

$$\Rightarrow \quad \sin\left(-\frac{\pi}{4}\right) \le a \le \sin\frac{\pi}{4}$$

$$\Rightarrow \quad -\frac{1}{\sqrt{2}} \le a \le \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad |a| \le \frac{1}{\sqrt{2}}$$

20 $\tan^{-1}\left(\frac{\tan x}{4}\right) + \tan^{-1}\left(\frac{3\sin 2x}{5 + 3\cos 2x}\right)$

$$= \tan^{-1}\left(\frac{\tan x}{4}\right) + \tan^{-1}\left(\frac{\frac{6\tan x}{1+\tan^2 x}}{5 + \frac{3(1-\tan^2 x)}{1+\tan^2 x}}\right)$$

$$= \tan^{-1}\left(\frac{\tan x}{4}\right) + \tan^{-1}\left(\frac{6\tan x}{8 + 2\tan^2 x}\right)$$

$$= \tan^{-1}\left(\frac{\tan x}{4}\right) + \tan^{-1}\left(\frac{3\tan x}{4 + \tan^2 x}\right)$$

$$= \tan^{-1}\left(\frac{\frac{\tan x}{4} + \frac{3\tan x}{4+\tan^2 x}}{1 - \frac{3\tan^2 x}{4(4+\tan^2 x)}}\right)$$

$$\left(\text{as } \left|\frac{\tan x}{4}\cdot\frac{3\tan x}{4\tan^2 x}\right| < 1\right)$$

$$= \tan^{-1}\left(\frac{16\tan x + \tan^3 x}{16 + \tan^2 x}\right)$$

$$= \tan^{-1}(\tan x) = x$$

21 $\tan\theta + \tan\left(\frac{\pi}{3}+\theta\right) + \tan\left(-\frac{\pi}{3}+\theta\right)$

$= a\tan 3\theta$

$\Rightarrow \tan\theta + \frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta}$

$+ \frac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta} = a\tan 3\theta$

$\Rightarrow \tan\theta + \frac{8\tan\theta}{1-3\tan^2\theta} = a\tan 3\theta$

$\Rightarrow \frac{3(3\tan\theta-\tan^3\theta)}{1-3\tan^2\theta} = a\tan 3\theta$

$\Rightarrow 3\tan 3\theta = a\tan 3\theta$

$\Rightarrow a = 3$

22 Let $\cos^{-1}x = \tan^{-1}x = \theta$

$\Rightarrow x = \cos\theta = \tan\theta$

$\Rightarrow \cos\theta = \tan\theta \Rightarrow \cos\theta = \frac{\sin\theta}{\cos\theta}$

$\Rightarrow \cos^2\theta = \sin\theta$

$\Rightarrow \sin^2\theta + \sin\theta - 1 = 0$

$\Rightarrow \sin\theta = \frac{-1\pm\sqrt{1+4}}{2}$

$\Rightarrow \sin\theta = \frac{\sqrt{5}-1}{2}$

$\therefore x^2 = \cos^2\theta = \sin\theta = \frac{\sqrt{5}-1}{2}$

and $\sin(\cos^{-1}x) = \sin\theta$

$= \frac{\sqrt{5}-1}{2} = x^2$

23 Given, $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}\frac{1}{\sqrt{1+(x^2+x)}} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$

$\Rightarrow \cos^{-1}\frac{1}{\sqrt{1+(x^2+x)}} = \frac{\pi}{2} - \sin^{-1}\sqrt{x^2+x+1}$

$\Rightarrow \cos^{-1}\frac{1}{\sqrt{x^2+x+1}} = \cos^{-1}\sqrt{x^2+x+1}$

$\Rightarrow \frac{1}{\sqrt{x^2+x+1}} = \sqrt{x^2+x+1}$

$\Rightarrow x^2+x+1 = 1 \Rightarrow x = -1, 0$

24 Since, $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{4}{5}\right)$

$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right) = \cos^{-1}\left(\frac{4}{5}\right)$

$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right)$

$\therefore x = 3$

25 Given,

$\operatorname{cosec}^{-1}x + \cos^{-1}y + \sec^{-1}z \ge \alpha^2 - \sqrt{2\pi}\,\alpha + 3\pi$

$\text{RHS} = \alpha^2 - \sqrt{2\pi}\,\alpha + 3\pi$

$= \alpha^2 - 2\sqrt{\frac{\pi}{2}}\,\alpha + \frac{\pi}{2} + 3\pi - \frac{\pi}{2}$

$= \left(\alpha - \sqrt{\frac{\pi}{2}}\right)^2 + \frac{5\pi}{2} \ge \frac{5\pi}{2}$ ...(i)

$\because$ LHS, $\operatorname{cosec}^{-1}x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$\cos^{-1}y \in [0, \pi]$

and $\sec^{-1}z \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$\therefore \text{LHS} \le \frac{5\pi}{2}$ ...(ii)

From Eqs. (i) and (ii), we get only possibility is sign of equality

$x = 1, y = -1, z = -1$

26 Given, $\cos^{-1}x \ge \sin^{-1}x$

$\Rightarrow \frac{\pi}{2} \ge 2\sin^{-1}x$

$\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \forall x \in [-1,1]\right]$

$\Rightarrow \sin^{-1}x \le \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$

$\left(\because \text{range of } \sin^{-1}x \text{ is } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$

$\Rightarrow -1 \le x \le \sin\left(\frac{\pi}{4}\right)$

$\Rightarrow x \in \left[-1, \frac{1}{\sqrt{2}}\right]$

27 $\left(\frac{\pi}{2} - \sin^{-1}x\right)^2 + (\sin^{-1}x)^2$

$= \frac{\pi^2}{4} + 2(\sin^{-1}x)^2 - \pi\sin^{-1}x$

$= \frac{\pi^2}{8} + 2\left[\sin^{-1}x - \frac{\pi}{4}\right]^2$

Here, $m = \frac{\pi^2}{8}, M = \frac{5\pi^2}{4}$

$\therefore \frac{M}{m} = 10$

28 Given, $\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x)$

$\therefore \sqrt{2\cos^2 x} = \sqrt{2}\,x$

$\Rightarrow \sqrt{2}\,|\cos x| = \sqrt{2}\,x$

For $x \in \left[\frac{\pi}{2}, \pi\right]$, $|\cos x| = -\cos x$

$-\sqrt{2}\cos x = \sqrt{2}\,x$

$\Rightarrow -\cos x = x$

$\therefore \cos x = -x$

Hence, no solution exist.

29 $\because T_r = \sin^{-1}\left\{\frac{\sqrt{r}-\sqrt{(r-1)}}{\sqrt{r(r+1)}}\right\}$

$= \tan^{-1}\left\{\frac{\sqrt{r}-\sqrt{(r-1)}}{1+\sqrt{r}(\sqrt{r-1})}\right\}$

$\therefore S_n = \sum_{r=1}^{n}\tan^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{1+\sqrt{r}\sqrt{r-1}}\right)$

$= \sum_{r=1}^{n}[\tan^{-1}\sqrt{r} - \tan^{-1}\sqrt{(r-1)}]$

$= \tan^{-1}\sqrt{n} - \tan^{-1}\sqrt{0}$

$= \tan^{-1}\sqrt{n} - 0$

$\therefore S_\infty = \tan^{-1}\infty = \frac{\pi}{2}$

30 Now, $\tan\left\{2\tan^{-1}\left(\frac{1}{5}\right)\right\} = \frac{2\times\frac{1}{5}}{1-\frac{1}{25}} = \frac{5}{12}$

Given equation can be rewritten as

$17x^2 - 17x\tan\left\{\frac{\pi}{4} - 2\tan^{-1}\left(\frac{1}{5}\right)\right\} - 10 = 0$

$\Rightarrow 17x^2 - 17x\cdot\frac{1-\frac{5}{12}}{1+\frac{5}{12}} - 10 = 0$

$\Rightarrow 17x^2 - 7x - 10 = 0$

$\Rightarrow (x-1)(17x+10) = 0$

Hence, $x = 1$ is a root of the given equation.

31 $\sin\left(\sin^{-1}\frac{1}{\sqrt{(1+x)^2+1}}\right)$

$= \cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)$

$\frac{1}{\sqrt{(1+x)^2+1}} = \frac{1}{\sqrt{1+x^2}}$

$\Rightarrow (1+x)^2 + 1 = 1 + x^2$

$\Rightarrow 2x + 1 = 0$

$\Rightarrow x = -\frac{1}{2}$

32 We have, $0 < x < 1$

Now, $\sqrt{1+x^2}[\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1]^{1/2}$

$= \sqrt{1+x^2}\left[\left\{\frac{x\cdot x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}}\right\}^2 - 1\right]^{1/2}$

$= \sqrt{1+x^2}\left[\left(\frac{1+x^2}{\sqrt{1+x^2}}\right)^2 - 1\right]^{1/2}$

$= \sqrt{1+x^2}[1+x^2-1]^{1/2}$

$= x\sqrt{1+x^2}$

33 Since, x, y and z are in AP.

$\therefore \quad 2y = x + z$

Also, $\tan^{-1} x$, $\tan^{-1} y$ and $\tan^{-1} z$ are in AP.

$\therefore 2\tan^{-1} y = \tan^{-1} x + \tan^{-1} z$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2y}{1 - y^2}\right) = \tan^{-1}\left(\frac{x + z}{1 - xz}\right)$$

$$\Rightarrow \quad \frac{x + z}{1 - y^2} = \frac{x + z}{1 - xz}$$

$$\Rightarrow \quad y^2 = xz \qquad [\because 2y = x + z]$$

Since x, y and z are in AP as well as in GP.

$\therefore \quad x = y = z$

SESSION 2

1 Now, $x - \frac{x^2}{2} + \frac{x^3}{4} - \dots$

$$= \frac{x}{1 + \frac{x}{2}} = \frac{2x}{2 + x}$$

and $\quad x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \dots$

$$= \frac{x^2}{1 + \frac{x^2}{2}} = \frac{2x^2}{2 + x^2}$$

$$\because \sin^{-1} \alpha + \cos^{-1} \alpha = \frac{\pi}{2}$$

$$\therefore \quad \frac{2x}{2 + x} = \frac{2x^2}{2 + x^2}, x \neq 0$$

$$\Rightarrow \quad 2 + x^2 = 2x + x^2$$

$\therefore \quad x = 1$

2 $\because \quad f(x) = ax + b$

$\therefore f'(x) = a > 0$

So, $f(x)$ is an increasing function.

$\Rightarrow \quad f(-1) = 0$ and $f(1) = 2$

$\Rightarrow \quad -a + b = 0$

and $\quad a + b = 2$

Then, $\quad a = b = 1$

$\therefore \quad f(x) = x + 1$

Now, $\cot[\cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18]$

$$= \cot\left[\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right) + \tan^{-1}\left(\frac{1}{18}\right)\right]$$

$$= \cot\left[\tan^{-1}\left(\frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{7} \cdot \frac{1}{8}}\right) + \tan^{-1}\left(\frac{1}{18}\right)\right]$$

$$= \cot\left[\tan^{-1}\left(\frac{15}{55}\right) + \tan^{-1}\left(\frac{1}{18}\right)\right]$$

$$= \cot\left[\tan^{-1}\left(\frac{3}{11}\right) + \tan^{-1}\left(\frac{1}{18}\right)\right]$$

$$= \cot\left[\tan^{-1}\left(\frac{\frac{3}{11} + \frac{1}{18}}{1 - \frac{3}{11} \cdot \frac{1}{18}}\right)\right]$$

$$= \cot\left[\tan^{-1}\left(\frac{65}{195}\right)\right]$$

$$= \cot\left[\tan^{-1}\left(\frac{1}{3}\right)\right] = \cot(\cot^{-1} 3) = 3$$

$$= 1 + 2 = f(2) \qquad [\because f(x) = x + 1]$$

3 $$S = \tan^{-1}\left\{\frac{(n + 1) - (n + 0)}{1 + (n + 0)(n + 1)}\right\} + \tan^{-1}\left\{\frac{(n + 2) - (n + 1)}{1 + (n + 1)(n + 2)}\right\} + \dots + \tan^{-1}\left\{\frac{(n + 20) - (n + 19)}{1 + (n + 19)(n + 20)}\right\}$$

$$= \tan^{-1}(n + 1) - \tan^{-1} n + \tan^{-1}(n + 2) - \tan^{-1}(n + 1) + \dots + \tan^{-1}(n + 20) - \tan^{-1}(n + 19)$$

$$= \tan^{-1}(n + 20) - \tan^{-1} n$$

$$= \tan^{-1}\left\{\frac{n + 20 - n}{1 + n(n + 20)}\right\}$$

$$= \tan^{-1}\left(\frac{20}{n^2 + 20n + 1}\right)$$

$$\Rightarrow \tan S = \tan\left\{\tan^{-1}\left(\frac{20}{n^2 + 20n + 1}\right)\right\}$$

$$\Rightarrow \tan S = \frac{20}{n^2 + 20n + 1}$$

4 Given, $f(x) = e^{\cos^{-1} \sin\left(x + \frac{\pi}{3}\right)}$

$$\Rightarrow \quad f\left(\frac{8\pi}{9}\right) = e^{\cos^{-1} \sin\left(\frac{8\pi}{9} + \frac{\pi}{3}\right)} = e^{\cos^{-1} \sin\left(\frac{11\pi}{9}\right)}$$

$$\Rightarrow \quad f\left(\frac{8\pi}{9}\right) = e^{\cos^{-1} \cos\left(\frac{13\pi}{18}\right)} = e^{\frac{13\pi}{18}}$$

Also, $f\left(-\frac{7\pi}{4}\right) = e^{\cos^{-1} \sin\left(-\frac{7\pi}{4} + \frac{\pi}{3}\right)}$

$$= e^{\cos^{-1} \sin\left(-\frac{17\pi}{12}\right)} = e^{\cos^{-1} \cos\frac{\pi}{12}} = e^{\frac{\pi}{12}}$$

5 $\because -\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq \sin^{-1} y \leq \frac{\pi}{2}$

and $-\frac{\pi}{2} \leq \sin^{-1} z \leq \frac{\pi}{2}$

Given that,

$$\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$$

which is possible only when

$$\sin^{-1} x = \sin^{-1} y = \sin^{-1} z = \frac{\pi}{2}$$

$\Rightarrow \quad x = y = z = 1$

Put $\quad p = q = 1$

Then, $f(2) = f(1)f(1) = 2 \cdot 2 = 4$

and put $p = 1, q = 2$

Then, $f(3) = f(1)f(2) = 2 \cdot 2^2 = 8$

$$\therefore x^{f(1)} + y^{f(2)} + z^{f(3)} - \frac{x + y + z}{x^{f(1)} + y^{f(2)} + z^{f(3)}}$$

$$= 1 + 1 + 1 - \frac{3}{1 + 1 + 1} = 3 - 1 = 2$$

6 $\because \quad 0 \leq \cos^{-1} x \leq \pi$

and $0 < \cot^{-1} x < \pi$

Given, $[\cot^{-1} x] + [\cos^{-1} x] = 0$

$\Rightarrow \quad [\cot^{-1} x] = 0 \quad$ and $\quad [\cos^{-1} x] = 0$

$\Rightarrow 0 < \cot^{-1} x < 1$ and $0 \leq \cos^{-1} x < 1$

$\therefore \quad x \in (\cot 1, \infty)$

and $\quad x \in (\cos 1, 1) \Rightarrow x \in (\cot 1, 1)$

7 Given that,

$$\cot^{-1}(\sqrt{\cos \alpha}) - \tan^{-1}(\sqrt{\cos \alpha}) = x \quad \dots(i)$$

We know that,

$$\cot^{-1}(\sqrt{\cos \alpha}) + \tan^{-1}(\sqrt{\cos \alpha}) = \frac{\pi}{2} \; 3 \quad \dots(ii)$$

$$\left[\because \cot^{-1} x + \tan^{-1} x = \frac{\pi}{2}\right]$$

On adding Eqs. (i) and (ii), we get

$$2\cot^{-1}(\sqrt{\cos \alpha}) = \frac{\pi}{2} + x$$

$$\Rightarrow \quad \sqrt{\cos \alpha} = \cot\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$\Rightarrow \quad \sqrt{\cos \alpha} = \frac{\cot \frac{x}{2} - 1}{1 + \cot \frac{x}{2}}$$

$$\Rightarrow \quad \sqrt{\cos \alpha} = \frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2}}$$

On squaring both sides, we get

$$\Rightarrow \quad \cos \alpha = \frac{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} - 2\sin \frac{x}{2} \cos \frac{x}{2}}{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2\sin \frac{x}{2} \cos \frac{x}{2}}$$

$$\Rightarrow \quad \cos \alpha = \frac{1 - \sin x}{1 + \sin x}$$

$$\Rightarrow \quad \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} = \frac{1 - \sin x}{1 + \sin x}$$

On applying componendo and dividendo rule, we get

$$\sin x = \tan^2\left(\frac{\alpha}{2}\right)$$

8 Given,

$(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$

$\Rightarrow (\tan^{-1} x + \cot^{-1} x)^2 - 2\tan^{-1} x \cdot \cot^{-1} x = \frac{5\pi^2}{8}$

$\Rightarrow \left(\frac{\pi}{2}\right)^2 - 2\tan^{-1} x\left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{5\pi^2}{8} \quad \left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right]$

$\Rightarrow \frac{\pi^2}{4} - 2 \cdot \frac{\pi}{2} \tan^{-1} x + 2(\tan^{-1} x)^2 = \frac{5\pi^2}{8}$

$\Rightarrow 2(\tan^{-1} x)^2 - \pi \tan^{-1} x - \frac{3\pi^2}{8} = 0$

$\Rightarrow \tan^{-1} x = -\frac{\pi}{4}, \frac{3\pi}{4}$

$\Rightarrow \tan^{-1} x = -\frac{\pi}{4}$

$\Rightarrow x = -1$

$\left\{\text{neglecting } \tan^{-1} x = \frac{3\pi}{4} \text{ as principal value of } \tan^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$

9 $\tan^2(\sin^{-1} x) > 1$

$\Rightarrow \frac{\pi}{4} < \sin^{-1} x < \frac{\pi}{2}$

or $-\frac{\pi}{2} < \sin^{-1} x < -\frac{\pi}{4}$

$\Rightarrow x \in \left(\frac{1}{\sqrt{2}}, 1\right)$ or $x \in \left(-1, -\frac{1}{\sqrt{2}}\right)$

$\Rightarrow x \in \left(-1, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right)$

10 $8x^2 + 22x + 5 = 0$

$\Rightarrow x = -\frac{1}{4}, -\frac{5}{2}$

$\because -1 < -\frac{1}{4} < 1$ and $-\frac{5}{2} < -1$

$\therefore$ $\sin^{-1}\left(-\frac{1}{4}\right)$ exists but $\sin^{-1}\left(-\frac{5}{2}\right)$ does not exist.

$\sec^{-1}\left(-\frac{5}{2}\right)$ exists but $\sec^{-1}\left(-\frac{1}{4}\right)$ does not exist.

So, $\tan^{-1}\left(-\frac{1}{4}\right)$

and $\tan^{-1}\left(-\frac{5}{2}\right)$ both exist.

11 Let $\tan^{-1}(-2) = \theta$

$\Rightarrow \tan\theta = -2$

$\Rightarrow \theta \in \left(-\frac{\pi}{2}, 0\right)$

$\Rightarrow 2\theta \in (-\pi, 0)$

Now, $\cos(-2\theta) = \cos 2\theta$

$= \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$

$= \frac{-3}{5}$

$\Rightarrow -2\theta = \cos^{-1}\left(\frac{-3}{5}\right)$

$= \pi - \cos^{-1}\frac{3}{5}$

$\Rightarrow 2\theta = -\pi + \cos^{-1}\frac{3}{5}$

$\Rightarrow 2\theta = -\pi + \tan^{-1}\frac{4}{3}$

$= -\pi + \cot^{-1}\frac{3}{4}$

$= -\pi + \frac{\pi}{2} - \tan^{-1}\frac{3}{4}$

$= -\frac{\pi}{2} - \tan^{-1}\frac{3}{4}$

$= -\frac{\pi}{2} + \tan^{-1}\left(-\frac{3}{4}\right)$

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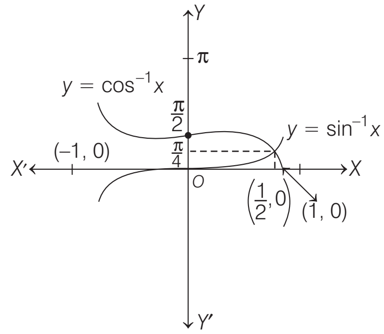

$\therefore \sin^{-1} x > \cos^{-1} x, \forall x \in \left(\frac{1}{\sqrt{2}}, 1\right)$